March 28, 2017

Wednesday, April 5, 2017 3:47 PM

It is known previously that

$$A \subset X \text{ is closed} \xRightarrow{X \text{ is compact}} A \text{ is compact}$$

Question

Any condition?

Before giving the answer, recall that

* continuous image of compact space is also compact

Example.

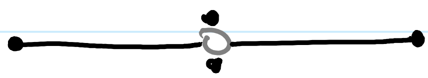
Consider $[-1,1] \sqcup [-1,1]$, obvious compact

$$\downarrow q$$

Thus

$$X = \left([-1,1] \sqcup [-1,1]\right)/\sim \text{ is compact}$$

$$\|$$



Moreover, $q([-1,1] \sqcup \phi)$ and $q(\phi \sqcup [-1,1])$ are also compact subsets of X

However, they are not closed.

$$A \text{ is closed} \;\cancel{\xLeftarrow{X \text{ is compact}}}\; A \subset X \text{ is compact}$$

03-28-p2

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Theorem. Let X be Hausdorff.
$A \subset X$ is compact $\Longrightarrow$ A is closed.

Corollary. Let X be compact Hausdorff.
$A \subset X$ is closed $\Longleftrightarrow$ A is compact.

Interestingly, the above theorem has a surprised effect leading to a useful result.

Theorem. Let X be compact and Y be Hausdorff.
Any continuous bijection $f: X \longrightarrow Y$ is a homeomorphism.

That is, continuity of f^{-1} is automatic.

Sketch of proof. To show f^{-1} is continuous

$$X \xleftarrow{\;f^{-1}\;} Y$$

$\cap$

Take any $U \in \mathcal{T}_X$ — Ask if its pre-image $(f^{-1})^{-1}(U) \in \mathcal{T}_Y$

$\rightsquigarrow$ Any closed $F = X \setminus U$ — $Y \setminus f(U)$

$\Downarrow$ X is compact — $\Uparrow$

F is compact $\xRightarrow{f \text{ is continuous}}$ $f(F)$ is compact $\xRightarrow{Y \text{ is } T_2}$ $f(F)$ is closed

03-28-p3

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Proof of Theorem: Given X Hausdorff, any compact subset $A \subset X$ will be closed.

Try to show $X \setminus A \in \mathcal{T}$

Take any $x \in X \setminus A$

$\vdots$

$\exists\, U \in \mathcal{T}$ such that $x \in \underbrace{U \subset X \setminus A}_{U \cap A = \emptyset}$

For any $a \in A$, clearly $x \neq a$,

As X is Hausdorff, $\exists\, U_a, V_a \in \mathcal{T}$

$x \in U_a$, $a \in V_a$, $U_a \cap V_a = \emptyset$

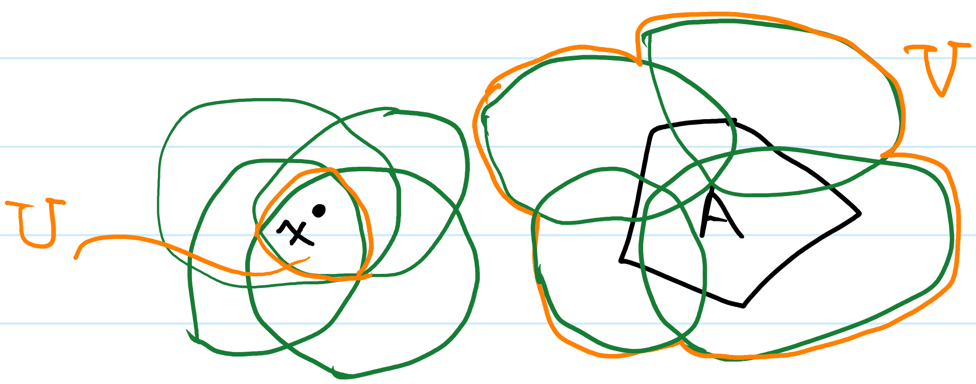
By this $\{V_a : a \in A\}$ is an open cover for A and has a finite subcover, $\{V_{a_1}, V_{a_2}, \cdots, V_{a_n}\}$.

Corresponding, $x \in U_{a_k}$, $k = 1, \cdots, n$

Define $U = \bigcap_{k=1}^{n} U_{a_k} \in \mathcal{T}$ and $V \hat{=} \bigcup_{k=1}^{n} V_{a_k} \in \mathcal{T}$

Then $x \in U$ and $A \subset V$ and $U \cap V = \emptyset$

In particular, $x \in U \subset X \setminus A$.



Actually proved: Let X be Hausdorff

$\forall$ compact $A \subset X$, $\forall\, x \notin A$

$\exists\, U, V \in \mathcal{T}$ such that

$x \in U,\ A \subset V,\ U \cap V = \emptyset$

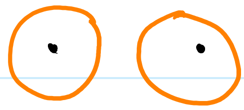
This looks very familiar! Compare it with Hausdorff (T_2):

$\forall\, x \neq y$, $\exists\, U, V \in \mathcal{T}$ such that

$x \in U,\ y \in V,\ U \cap V = \emptyset$

Separation Axioms

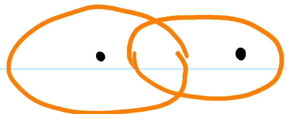
T_2: Hausdorff



T_3: T_1 + regular

T_1: $\forall\, x \neq y\ \exists\, U, V \in \mathcal{T}$

$x \in U \setminus V$ and $y \in V \setminus U$

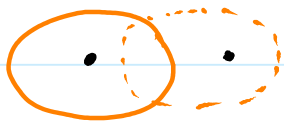


Equivalently, each singleton is closed

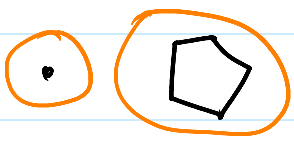
T_4: T_1 + normal

T_0: $\forall\, x \neq y\ \exists\, U \in \mathcal{T}$ such that

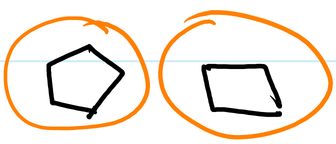
$x \in U$ & $y \notin U$ or $x \notin U$ & $y \in U$



Regular:



Normal:



Definition. A space $(X, \mathcal{T})$ is

* **regular** if $\forall$ closed $F \subset X$, $\forall\, x \notin F$
  $\exists\, U, V \in \mathcal{T}$ such that
  $x \in U$, $F \subset V$, $U \cap V = \emptyset$
* **normal** if $\forall$ closed $E, F \subset X$ with $E \cap F = \emptyset$
  $\exists\, U, V \in \mathcal{T}$ such that
  $E \subset U$, $F \subset V$, $U \cap V = \emptyset$.
* There are other notions such as completely regular ($T_{3.5}$) or perfectly regular (T_6)

Observations. In a compact space X, closed subsets are compact. By the theorem above, if X is compact Hausdorff, then a closed set (thus compact) and a point can be separated by open sets, i.e., X is regular. Then one may further use the same technique to prove that two disjoint closed sets can be separated by open sets, i.e., X is normal.

Theorem. A compact Hausdorff space is indeed T_3 and T_4.

Advantage of Regular

Let us consider a normal space X.

Take any $x \in U$ where $U \in \mathcal{T}$.

Then $x \notin X \setminus U$ and $X \setminus U$ is closed.

As X is regular, $\exists\ U_1, V_1 \in \mathcal{T}$

$$x \in U_1 \ , \ \underbrace{X \setminus U \subset V_1}_{U \supset X \setminus V_1} \ , \text{ and } \underbrace{U_1 \cap V_1 = \phi}_{X \setminus V_1 \supset U_1}$$

Since $X \setminus V_1$ is closed, $X \setminus V_1 \supset \overline{U_1}$

So, we have $x \in U_1 \subset \overline{U_1} \subset U$

Iteratively, we have a chain

$$x \in \cdots U_3 \subset \overline{U_3} \subset U_2 \subset \overline{U_2} \subset U_1 \subset \overline{U_1} \subset U$$

Urysohn Lemma. Let X be a normal space.
If $A, B \subset X$ are closed, $A \cap B = \phi$
then $\exists$ continuous $f : X \longrightarrow [-1, 1]$ such that

$$f|_A \equiv -1 \quad \text{and} \quad f|_B \equiv 1.$$

This is a major tool to derive the **Tietz Extension Theorem.**